6-6 Solving Polynomial Equations

Main Ideas

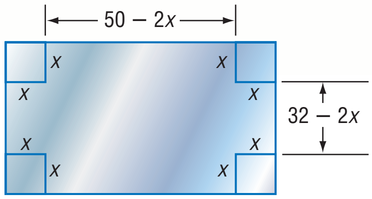
- Factor polynomials.
- Solve polynomial equations by factoring.

New Vocabulary

quadratic form

GET READY for the Lesson

The Taylor Manufacturing Company makes open metal boxes of various sizes. Each sheet of metal is 50 inches long and 32 inches wide. To make a box, a square is cut from each corner.



The volume of the box depends on the side length x of the cut squares. It is given by $V(x) = 4x^3 - 164x^2 + 1600x$. You can solve a polynomial equation to find the dimensions of the square to cut for a box with specific volume.

Factor Polynomials Whole numbers are factored using prime numbers. For example, $100 = 2 \cdot 2 \cdot 5 \cdot 5$. Many polynomials can also be factored. Their factors, however, are other polynomials. Polynomials that cannot be factored are called *prime*. One method for finding the dimensions of the square to cut to make a box involves factoring the polynomial that represents the volume.

The table below summarizes the most common factoring techniques used with polynomials. Some of these techniques were introduced in Lesson 5-3. The others will be presented in this lesson.

CONCEPT SUMMARY *Factoring Techniques*

Number of Terms	Factoring Technique	General Case
any number	Greatest Common Factor (GCF)	$a^3b^2 + 2a^2b - 4ab^2 = ab(a^2b + 2a - 4b)$
two	Difference of Two Squares Sum of Two Cubes Difference of Two Cubes	$a^2 - b^2 = (a + b)(a - b)$ $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
three	Perfect Square Trinomials	$a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$
	General Trinomials	$acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$
four or more	Grouping	$ax + bx + ay + by = x(a + b) + y(a + b)$ $= (a + b)(x + y)$

Whenever you factor a polynomial, always look for a common factor first. Then determine whether the resulting polynomial factor can be factored again using one or more of the methods listed above.

Study Tip

Checking You can check the result when factoring by finding the product.

EXAMPLE 1 GCF

Factor $6x^2y^2 - 2xy^2 + 6x^3y$.

$$6x^2y^2 - 2xy^2 + 6x^3y = (2 \cdot 3 \cdot x \cdot x \cdot y \cdot y) - (2 \cdot x \cdot y \cdot y) + (2 \cdot 3 \cdot x \cdot x \cdot x \cdot y)$$

$= (2xy \cdot 3xy) - (2xy \cdot y) + (2xy \cdot 3x^2)$ The GCF is $2xy$. The remaining polynomial cannot be factored using the methods above.

$= 2xy(3xy - y + 3x^2)$

CHECK Your Progress **Factor completely.**

1A. $18x^3y^4 + 12x^2y^3 - 6xy^2$ **1B.** $a^4b^4 + 3a^3b^4 + a^2b^3$

EXAMPLE 2 Grouping

Factor $a^3 - 4a^2 + 3a - 12$.

$a^3 - 4a^2 + 3a - 12 = (a^3 - 4a^2) + (3a - 12)$ Group to find a GCF.

$= a^2(a - 4) + 3(a - 4)$ Factor the GCF of each binomial.

$= (a - 4)(a^2 + 3)$ Distributive Property

CHECK Your Progress **Factor completely.**

2A. $x^2 + 3xy + 2xy^2 + 6y^3$ **2B.** $6a^3 - 9a^2b + 4ab - 6b^2$

Factoring by grouping is the only method that can be used to factor polynomials with four or more terms. For polynomials with two or three terms, it may be possible to factor the polynomial according to one of the patterns shown on page 349.

EXAMPLE 3 Two or Three Terms

Factor each polynomial.

a. $8x^3 - 24x^2 + 18x$

This trinomial does not fit any of the factoring patterns. First, factor out the GCF. Then the remaining trinomial is a perfect square trinomial.

$8x^3 - 24x^2 + 18x = 2x(4x^2 - 12x + 9)$ Factor out the GCF.

$= 2x(2x - 3)^2$ Perfect square trinomial

b. $m^6 - n^6$

This polynomial could be considered the difference of two squares or the difference of two cubes. The difference of two squares should always be done before the difference of two cubes. This will make the next step of the factorization easier.

$m^6 - n^6 = (m^3 + n^3)(m^3 - n^3)$ Difference of two squares

$= (m + n)(m^2 - mn + n^2)(m - n)(m^2 + mn + n^2)$ Sum and difference of two cubes

CHECK Your Progress

3A. $3xy^2 - 48x$ **3B.** $c^3d^3 + 27$

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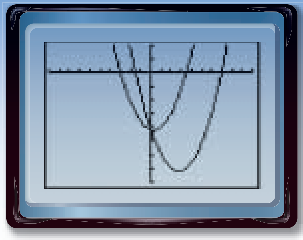
You can use a graphing calculator to check that the factored form of a polynomial is correct.

GRAPHING CALCULATOR LAB

Factoring Polynomials

Is the factored form of $2x^2 - 11x - 21$ equal to $(2x - 7)(x + 3)$? You can find out by graphing $y = 2x^2 - 11x - 21$ and $y = (2x - 7)(x + 3)$. If the two graphs coincide, the factored form is probably correct.

- Enter $y = 2x^2 - 11x - 21$ and $y = (2x - 7)(x + 3)$ on the **Y=** screen.
- Graph the functions. Since two different graphs appear, $2x^2 - 11x - 21 \neq (2x - 7)(x + 3)$.



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THINK AND DISCUSS

1. Determine if $x^2 + 5x - 6 = (x - 3)(x - 2)$ is a true statement. If not, write the correct factorization.
2. Does this method guarantee a way to check the factored form of a polynomial? Why or why not?

In some cases, you can rewrite a polynomial in x in the form $au^2 + bu + c$. For example, by letting $u = x^2$ the expression $x^4 - 16x^2 + 60$ can be written as $(x^2)^2 - 16(x^2) + 60$ or $u^2 - 16u + 60$. This new, but equivalent, expression is said to be in **quadratic form.**

KEY CONCEPT *Quadratic Form*

An expression that is quadratic in form can be written as $au^2 + bu + c$ for any numbers a, b, and c, $a \neq 0$, where u is some expression in x. The expression $au^2 + bu + c$ is called the quadratic form of the original expression.

EXAMPLE **Write Expressions in Quadratic Form**

4 **Write each expression in quadratic form, if possible.**

a. $x^4 + 13x^2 + 36$

$x^4 + 13x^2 + 36 = (x^2)^2 + 13(x^2) + 36$ $\quad (x^2)^2 = x^4$

b. $12x^8 - x^2 + 10$

This cannot be written in quadratic form since $x^8 \neq (x^2)^2$.

CHECK Your Progress

4A. $16x^6 - 625$

4B. $9x^{10} - 15x^4 + 9$

Solve Equations Using Quadratic Form In Chapter 5, you learned to solve quadratic equations by factoring and using the Zero Product Property. You can extend these techniques to solve higher-degree polynomial equations.

Study Tip

Substitution

To avoid confusion, you can substitute another variable for the expression in parentheses. For example, $x^4 - 13x^2 + 36 = 0$ could be written as $u^2 - 13u + 36 = 0$. Then once you have solved the equation for u, substitute x^2 for u and solve for x.

EXAMPLE Solve Polynomial Equations

5 Solve each equation.

a. $x^4 - 13x^2 + 36 = 0$

$x^4 - 13x^2 + 36 = 0$	Original equation
$(x^2)^2 - 13(x^2) + 36 = 0$	Write the expression on the left in quadratic form.
$(x^2 - 9)(x^2 - 4) = 0$	Factor the trinomial.
$(x - 3)(x + 3)(x - 2)(x + 2) = 0$	Factor each difference of squares.

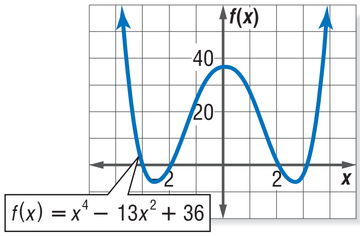
Use the Zero Product Property.

$x - 3 = 0$ or $x + 3 = 0$ or $x - 2 = 0$ or $x + 2 = 0$

$x = 3$ $\quad$ $x = -3$ $\quad$ $x = 2$ $\quad$ $x = -2$

The solutions are $-3, -2, 2,$ and 3.

CHECK The graph of $f(x) = x^4 - 13x^2 + 36$ shows that the graph intersects the x-axis at -3, $-2, 2,$ and 3.



b. $x^3 + 343 = 0$

$x^3 + 343 = 0$	Original equation
$(x)^3 + 7^3 = 0$	This is the sum of two cubes.
$(x + 7)[x^2 - x(7) + 7^2] = 0$	Sum of two cubes formula with $a = x$ and $b = 7$
$(x + 7)(x^2 - 7x + 49) = 0$	Simplify.
$(x + 7) = 0$ or $x^2 - 7x + 49 = 0$	Zero Product Property

The solution of the first equation is -7. The second equation can be solved by using the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

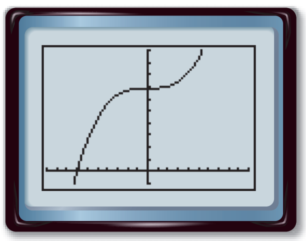
$$= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(49)}}{2(1)} \qquad \text{Replace } a \text{ with 1, } b \text{ with } -7\text{, and } c \text{ with 49.}$$

$$= \frac{7 \pm \sqrt{-147}}{2} \qquad \text{Simplify.}$$

$$= \frac{7 \pm i\sqrt{147}}{2} \text{ or } \frac{7 \pm 7i\sqrt{3}}{2} \qquad \sqrt{147} \times \sqrt{-1} = 7i\sqrt{3}$$

Thus, the solutions of the original equation are -7, $\frac{7 + 7i\sqrt{3}}{2}$, and $\frac{7 - 7i\sqrt{3}}{2}$.

CHECK The graph of $f(x) = x^3 + 343$ confirms the solution.



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CHECK Your Progress

5A. $x^4 - 29x^2 + 100 = 0$ $\qquad$ **5B.** $x^3 + 8 = 0$

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CHECK Your Understanding

Examples 1–3 (p. 350)

Factor completely. If the polynomial is not factorable, write *prime*.

1. $-12x^2 - 6x$

2. $a^2 + 5a + ab$

3. $21 - 7y + 3x - xy$

4. $y^2 + 4y + 2y + 8$

5. $z^2 - 4z - 12$

6. $3b^2 - 48$

7. $16w^2 - 169$

8. $h^3 + 8000$

Example 4 (p. 351)

Write each expression in quadratic form, if possible.

9. $5y^4 + 7y^3 - 8$

10. $84n^4 - 62n^2$

Example 5 (p. 352)

Solve each equation.

11. $x^4 - 50x^2 + 49 = 0$

12. $x^3 - 125 = 0$

13. POOL The Shelby University swimming pool is in the shape of a rectangular prism and has a volume of 28,000 cubic feet. The dimensions of the pool are x feet deep by $7x - 6$ feet wide by $9x - 2$ feet long. How deep is the pool?

Exercises

HOMEWORK HELP

For Exercises	See Examples
14–17	1
18, 19	2
20–23	3
24–29	4
30–39	5

Factor completely. If the polynomial is not factorable, write *prime*.

14. $2xy^3 - 10x$

15. $6a^2b^2 + 18ab^3$

16. $12cd^3 - 8c^2d^2 + 10c^5d^3$

17. $3a^2bx + 15cx^2y + 25ad^3y$

18. $8yz - 6z - 12y + 9$

19. $3ax - 15a + x - 5$

20. $y^2 - 5y + 4$

21. $2b^2 + 13b - 7$

22. $z^3 + 125$

23. $t^3 - 8$

Write each expression in quadratic form, if possible.

24. $2x^4 + 6x^2 - 10$

25. $a^8 + 10a^2 - 16$

26. $11n^6 + 44n^3$

27. $7b^5 - 4b^3 + 2b$

28. $7x^{\frac{1}{9}} - 3x^{\frac{1}{3}} + 4$

29. $6x^{\frac{2}{5}} - 4x^{\frac{1}{5}} - 16$

Solve each equation.

30. $x^4 - 34x^2 + 225 = 0$

31. $x^4 - 15x^2 - 16 = 0$

32. $x^4 + 6x^2 - 27 = 0$

33. $x^3 + 64 = 0$

34. $27x^3 + 1 = 0$

35. $8x^3 - 27 = 0$

Real-World Career

Designer

Designers combine practical knowledge with artistic ability to turn abstract ideas into formal designs.

Math Online

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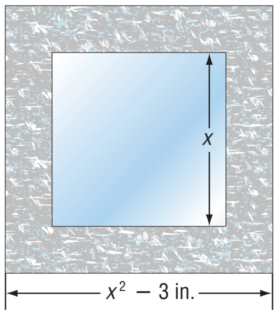
DESIGN For Exercises 36–38, use the following information.

Jill is designing a picture frame for an art project. She plans to have a square piece of glass in the center and surround it with a decorated ceramic frame, which will also be a square. The dimensions of the glass and frame are shown in the diagram at the right. Jill determines that she needs 27 square inches of material for the frame.



36. Write a polynomial equation that models the area of the frame.

37. What are the dimensions of the glass piece?

38. What are the dimensions of the frame?

39. **GEOMETRY** The width of a rectangular prism is w centimeters. The height is 2 centimeters less than the width. The length is 4 centimeters more than the width. If the volume of the prism is 8 times the measure of the length, find the dimensions of the prism.

40. Find the factorization of $3x^2 + x - 2$.

41. What are the factors of $2y^2 + 9y + 4$?

Factor completely. If the polynomial is not factorable, write prime.

42. $3n^2 + 21n - 24$

43. $y^4 - z^2$

44. $16a^2 + 25b^2$

45. $3x^2 - 27y^2$

46. $x^4 - 81$

47. $3a^3 + 2a^2 - 5a + 9a^2b + 6ab - 15b$

PACKAGING For Exercises 48 and 49, use the following information.

A computer manufacturer needs to change the dimensions of its foam packaging for a new model of computer. The width of the original piece is three times the height, and the length is equal to the height squared. The volume of the new piece can be represented by the equation $V(h) = 3h^4 + 11h^3 + 18h^2 + 44h + 24$, where h is the height of the original piece.

48. Factor the equation for the volume of the new piece to determine three expressions that represent the height, length, and width of the new piece.

49. How much did each dimension of the packaging increase for the new foam piece?

EXTRA PRACTICE
See pages 904, 931.

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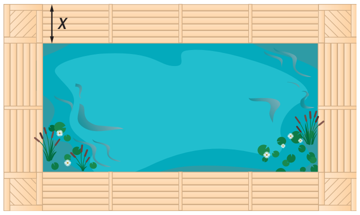
50. **LANDSCAPING** A boardwalk that is x feet wide is built around a rectangular pond. The pond is 30 feet wide and 40 feet long. The combined area of the pond and the boardwalk is 2000 square feet. What is the width of the boardwalk?



Graphing Calculator

CHECK FACTORING Use a graphing calculator to determine if each polynomial is factored correctly. Write *yes* or *no*. If the polynomial is not factored correctly, find the correct factorization.

51. $3x^2 + 5x + 2 \stackrel{?}{=} (3x + 2)(x + 1)$

52. $x^3 + 8 \stackrel{?}{=} (x + 2)(x^2 - x + 4)$

53. $2x^2 - 5x - 3 \stackrel{?}{=} (x - 1)(2x + 3)$

54. $3x^2 - 48 \stackrel{?}{=} 3(x + 4)(x - 4)$

H.O.T. Problems

55. **OPEN ENDED** Give an example of an equation that is not quadratic but can be written in quadratic form. Then write it in quadratic form.

56. **CHALLENGE** Factor $64p^{2n} + 16p^n + 1$.

57. **REASONING** Find a counterexample to the statement $a^2 + b^2 = (a + b)^2$.

58. **CHALLENGE** Explain how you would solve $|a - 3|^2 - 9|a - 3| = -8$. Then solve the equation.

59. ***Writing in Math*** Use the information on page 349 to explain how solving a polynomial equation can help you find dimensions. Explain how you could determine the dimensions of the cut square if the desired volume was 3600 cubic inches. Explain why there can be more than one square that can be cut to produce the same volume.

STANDARDIZED TEST PRACTICE

60. **ACT/SAT** Which is *not* a factor of $x^3 - x^2 - 2x$?

A x

B $x + 1$

C $x - 1$

D $x - 2$

61. **ACT/SAT** The measure of the largest angle of a triangle is 14 less than twice the measure of the smallest angle. The third angle is 2 more than the measure of the smallest angle. What is the measure of the smallest angle?

F 46 G 48 H 50 J 82

62. **REVIEW** $27x^3 + y^3 =$

A $(3x + y)(3x + y)(3x + y)$

B $(3x + y)(9x^2 - 3xy + y^2)$

C $(3x - y)(9x^2 + 3xy + y^2)$

D $(3x - y)(9x^2 + 9xy + y^2)$

Spiral Review

Graph each polynomial function. Estimate the x-coordinates at which the relative maxima and relative minima occur. (Lesson 6-5)

63. $f(x) = x^3 - 6x^2 + 4x + 3$

64. $f(x) = -x^4 + 2x^3 + 3x^2 - 7x + 4$

Find $p(7)$ and $p(-3)$ for each function. (Lesson 6-4)

65. $p(x) = x^2 - 5x + 3$

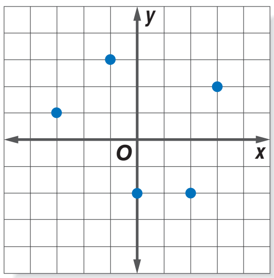
66. $p(x) = x^3 - 11x - 4$

67. $p(x) = \frac{2}{3}x^4 - 3x^3$

68. **PHOTOGRAPHY** The perimeter of a rectangular picture is 86 inches. Twice the width exceeds the length by 2 inches. What are the dimensions of the picture? (Lesson 3-2)

Determine whether each relation is a function. Write *yes* or *no*. (Lesson 2-1)

69.



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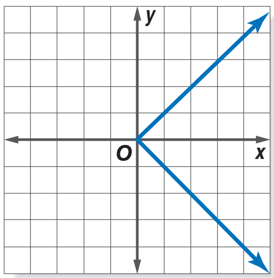

GET READY for the Next Lesson

PREREQUISITE SKILL **Find each quotient.** (Lesson 6-3)

71. $(x^3 + 4x^2 - 9x + 4) \div (x - 1)$

72. $(4x^3 - 8x^2 - 5x - 10) \div (x + 2)$

73. $(x^4 - 9x^2 - 2x + 6) \div (x - 3)$

74. $(x^4 + 3x^3 - 8x^2 + 5x - 6) \div (x + 1)$